Eur. J. Mech. B/Fluids

ON THE KOLMOGOROV AND FROZEN TURBULENCE IN NUMERICAL SIMULATION OF CAPILLARY WAVES

A. N. Pushkarev[1] (1)

[1]: *Arizona Center of Mathematical Sciences, University of Arizona, Tucson, AZ 85721, USA e-mail:* andrei@acms.arizona.edu

(Received 15 September 1998, revised and accepted 15 January 1999)

Abstract – Numerical simulation of dynamical equations for capillary waves excited by long-scale forcing shows the presence of both Kolmogorov spectrum at high wavenumbers (with the index predicted by weak-turbulent theory) and non-monotonic spectrum at low wavenumbers. The value of the Kolmogorov constant measured in numerical experiments happens to be different from the theoretical one. We explain the difference by the coexistence of Kolmogorov and "frozen" turbulence with the help of maps of quasi-resonances. Observed results are believed to be generic for different physical dispersive systems and are confirmed by laboratory experiments. © Elsevier, Paris

1. Introduction

An important issue is the degree of correspondence of wave dynamics in nonlinear dispersive media of infinite and finite extent. This question becomes especially important when the characteristic wavelength of the wave ensemble is comparable with the length of the finite domain and excited waves are weakly nonlinear. Such questions arise during attempts of confirmation of analitical results obtained for dynamical system in infinite domain by numerical simulation of the same dynamical system in the finite region with particular type of boundary conditions. Popular boundary conditions are periodical ones and the following consideration will deal with this particular case of boundary conditions. The result is believed to be true, however, for other types of boundary conditions, such as, zero of the function or its derivative, or their combination, etc.

It is well known that in the limit of low levels of excitations the behavior of an ensemble of waves in nonlinear dispersive systems in infinite domain is described by the kinetic equation for waves [1] and is known as "weak" turbulence. One of the classical examples of the weak turbulence is the system of weakly nonlinear capillary waves on the surface of deep water. One should emphasize that above question of degree of correspondence of the wave dynamics in infinite and finite domain is not only about comparison of numerical simulation and analysis, but also about the role of boundary conditions in laboratory experiments on excitation of capillary waves in finite-size containers [2].

An interaction of the Fourier modes in kinetic equation for capillary waves is performed through the interaction of triplets of waves which are solution of the system of equations usually referred by "conservation laws" or "resonances" [1]

$$\omega_{\vec{k_1}} + \omega_{\vec{k_2}} - \omega_{\vec{k_3}} = 0 \qquad (1)$$

$$\vec{k_1} + \vec{k_2} = \vec{k_3} \qquad (2)$$

(1) Permanent address: Landau Institute for Theoretical Physics, Russian Academy of Sciences, Russia 117940 GSP-1 Moscow V-334, ul.Kosygina 2

where $\vec{k}$ and $\omega_{\vec{k}}$ are the corresponding wave vector and frequency.

The system (1)–(2) always has solutions in the case of continuous spectrum for dispersion relation of capillary waves

$$\omega_{\vec{k}} = \sigma^{\frac{1}{2}} k^{\frac{3}{2}} \tag{3}$$

known as "decay-type" dispersion relation [1]; σ is the coefficient of surface tension. The situation changes, however, in the case of a finite domain. Fourier harmonics of the system with periodic boundary conditions are not continuous functions of the wavenumber anymore, like in the case of infinite domain, but infinite set of values defined at discrete equidistant wavenumbers. The question of existence of solution of the system (1)–(2) turns into, generally speaking, nontrivial number theory problem. Significant breakthrough in classification of existence of solutions of this system for different types of dispersion relations $\omega_{\vec{k}}$ was done by E.Kartashova [3]. It was shown, in particular, that the system (1)–(2) does not have solutions in the case of capillary waves dispersion relation (3) which means that there are no interacting Fourier modes in the kinetic equation for waves in the finite domain in this case.

The situation changes, however, if nonlinear dispersion correction δ_k due to finite amplitude of the excited wave is taken into account and capillary wave frequency becomes

$$\omega_k = \sigma^{\frac{1}{2}} k^{\frac{3}{2}} + \delta_k \tag{4}$$

Conservation laws (1)–(2) are transformed therefore into "quasi-conservation laws" or "quasi-resonances"

$$\omega_{k_1} + \omega_{k_2} - \omega_{k_3} = \Delta_{k_1 k_2 k_3} \tag{5}$$
$$\vec{k_1} + \vec{k_2} = \vec{k_3} \tag{6}$$
$$\Delta_{k_1 k_2 k_3} = \delta_{k_3} - \delta_{k_1} - \delta_{k_2} \tag{7}$$

It is clear that the system (5)–(7) has more degrees of freedom than the system (1)–(2) for solutions to exist due to the parameter $\Delta_{k_1 k_2 k_3}$ which measures the level of excitation of oscillations.

Below we outline the results of direct numerical simulation of the dynamical equations for capillary waves in periodical domain which shows that besides classical Kolmogorov turbulence regime exists another regime of "frozen" turbulence. The "frozen" turbulence regime is explained with the help of "maps" of "quaiziresonances" (5)–(7) which show "allowed" and "prohibited" Fourier modes as a result of the discreteness of the Fourier spectrum due to the periodicity of the boundary conditions.

2. Numerical model

We carried out numerical simulations of weakly nonlinear capillary waves in two-dimensional periodical domain based on discretization of the system of dynamical equation of surface waves on deep water [4] supplied with forcing $F_{\vec{r}}$ and damping terms $D_{\vec{r}}$:

$$\begin{aligned}
\frac{\partial \eta_{\vec{r}}}{\partial t} &= \left[|\hat{k}|\psi\right]_{\vec{r}} - \mathrm{div}(\eta\nabla\psi) - |\hat{k}|\left[\left[|\hat{k}|\psi\right]_{\vec{r}} \times \eta_{\vec{r}}\right]_{\vec{r}} + |\hat{k}|\left[|\hat{k}|\left[\left[|\hat{k}|\psi\right]_{\vec{r}} \times \eta_{\vec{r}}\right]_{\vec{r}} \times \eta_{\vec{r}}\right]_{\vec{r}} + \\
&+ \frac{1}{2}\Delta_{\vec{r}}\left[\left[|\hat{k}|\psi\right]_{\vec{r}} \times \eta_{\vec{r}}^2\right]_{\vec{r}} + \frac{1}{2}|\hat{k}|\left[\Delta_{\vec{r}}\psi \times \eta_{\vec{r}}^2\right]
\end{aligned} \tag{8}$$

$$\begin{aligned}
\frac{\partial \psi_{\vec{r}}}{\partial t} &= \sigma \mathrm{div}\frac{\nabla\eta}{\sqrt{1+(\nabla\eta)^2}} + \frac{1}{2}\left[-(\nabla\psi)^2 + \left[|\hat{k}|\psi\right]_{\vec{r}}^2\right] \\
&-|\hat{k}|\left[\left[|\hat{k}|\psi\right]_{\vec{r}} \times \eta_{\vec{r}}\right]_{\vec{r}} \times \left[|\hat{k}|\psi\right]_{\vec{r}} - \Delta\psi \times \left[|\hat{k}|\psi\right]_{\vec{r}} \times \eta_{\vec{r}} + D_{\vec{r}} + F_{\vec{r}}
\end{aligned} \tag{9}$$

where $\eta_{\vec{r}} = \eta(\vec{r}, t)$ is the shape of the surface, $\vec{r} = (x, y)$; $\psi(\vec{r}, t)$ is the velocity potential evaluated on the free surface.

Brackets $[...]_{\vec{r}}$ denote an expression in R-space and action of the operator $|\hat{k}|$ on the function $\psi_{\vec{r}}$ is defined through Fourier space as

$$\left[|\hat{k}|\psi\right]_{\vec{r}} = \frac{1}{2\pi} \int |k| \psi_{\vec{k}} e^{i\vec{k}\vec{r}} d\vec{k}$$

The Fourier component of forcing is defined by

$$F_{\vec{k}} = f_{\vec{k}} \cos\left(\omega_k \left(1 + R\right) t\right) \tag{10}$$

where ω_k is the local linear frequency, $R = R(t)$ is a function of time, taking values randomly distributed between -1 and $+1$.

For numerical reasons we used an artificial damping D turned on at the wavenumbers $k_0 = 0.7 \div 0.9\, k_{max}$ (the details are not significant for the present paper and can be found in [4]).

In the absence of forcing and damping the system (8)–(9) preserves the energy integral H (Hamiltonian)

$$\begin{aligned}
H &= H_0 + H_1 + H_2 + ... \\
H_0 &= \frac{1}{2} \int \left[|k| |\psi_{\vec{k}}|^2 + (g + \sigma |k|^2) |\eta_{\vec{k}}|^2 \right] d\vec{k} \\
H_1 &= -\frac{1}{2} \frac{1}{2\pi} \int L_{\vec{k}_1 \vec{k}_2} \psi_{\vec{k}_1} \psi_{\vec{k}_2} \eta_{\vec{k}_3} \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 \\
H_2 &= \frac{1}{4(2\pi)^2} \int M_{\vec{k}\vec{k}_1\vec{k}_2\vec{k}_3} \psi_{\vec{k}} \psi_{\vec{k}_1} \eta_{\vec{k}_2} \eta_{\vec{k}_3} \delta(\vec{k} + \vec{k}_1 + \vec{k}_2 + \vec{k}_3) d\vec{k} d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 \\
L_{\vec{k}_1 \vec{k}_2} &= \vec{k}_1 \vec{k}_2 + |\vec{k}_1| |\vec{k}_2| \\
M_{\vec{k}_1 \vec{k}_2 \vec{k}_3 \vec{k}_4} &= |\vec{k}_1| |\vec{k}_2| \left[\frac{1}{2} \left[|\vec{k}_1 + \vec{k}_3| + |\vec{k}_1 + \vec{k}_4| + |\vec{k}_2 + \vec{k}_3| + |\vec{k}_2 + \vec{k}_4| \right] - |\vec{k}_1| - |\vec{k}_2| \right]
\end{aligned}$$

It is convenient to use the parameter $\epsilon = \frac{H_1 + H_2}{H_0}$ as a measure of nonlinearity of the system which by the order of magnitude is equal to the average angle of deviation of the liquid surface from the horizontal line $|\nabla \eta|$.

3. Kolmogorov and "frozen" turbulence

A series of experiments was carried out with the forcing (10) localized at small wavenumbers. They show that, at low levels of nonlinearity $\epsilon \simeq 10^{-2}$, the stationary spectrum of surface elevations is isotropic in angle and transfers a finite energy flux to the large wavenumbers $\vec{k}$ region.

The plot of the logarithmic derivative (see Fig. 1) shows that in the interval $8 < k < 23$ the spectrum can be considered as powerlike $I_k = qk^{-x}$. The exponential value is close to $x \simeq 4.8$ with $q \simeq 0.03$. This exponential value is in a good agreement with the value of Kolmogorov index $\frac{19}{4}$ calculated by Zakharov and Filonenko [5], [6] as an exact solution of stationary kinetic equations for waves.

From weak-turbulence theory, $q = C_{exp}\sqrt{P}$ $(\sigma = 1)$, where C_{exp} is an experimental value of the Kolmogorov constant. Once we have measured the energy flux P, one can calculate $C_{exp} = 1.7$ which happens to be different from the theoretical value of Kolmogorov constant $C_{theory} = 9.85$ [4].

Another series of experiments carried out for lower levels of nonlinearity, $\frac{H_1 + H_2}{H_0} \leq 10^{-3}$, has shown that there is a stationary regime of "frozen" turbulence in the small-wavenumbers region of pumping, with the spectrum exponentially decaying toward high $\vec{k}$ (see Fig.2). The wave spectrum consists of several dozens of excited low-wavenumber harmonics, possibly exchanging energy between each other, without generating an energy cascade

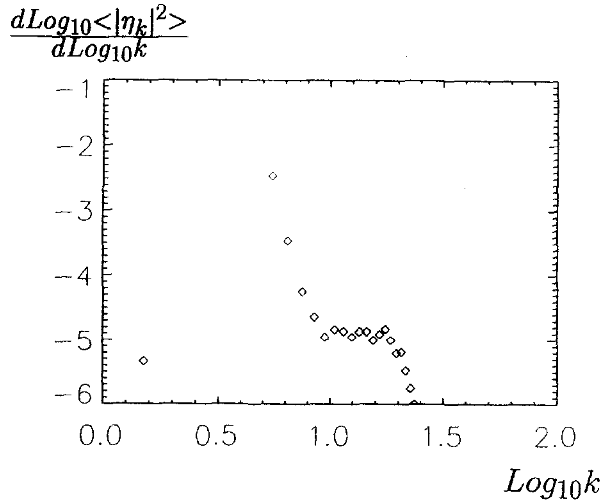

FIGURE 1. The derivative of the logarithm of the spectrum of spatial elevations with respect to the logarithm of the wavenumber, as a function of the logarithm of the wave number (local value of Kolmogorov index).

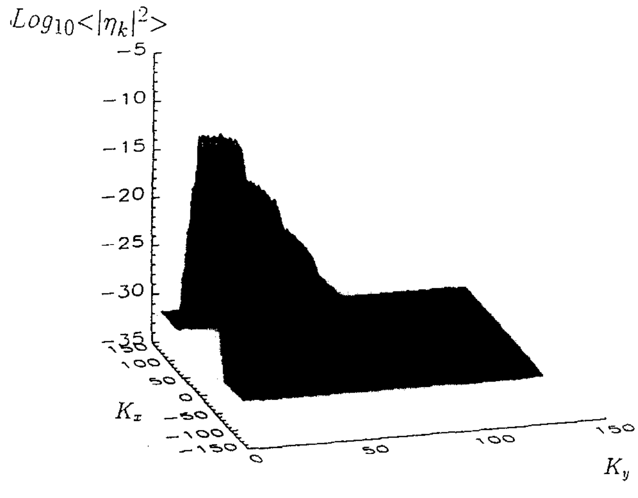

FIGURE 2. One half of the spectrum of spatial elevation in the case of "frozen" turbulence

toward high wavenumbers. There is virtually no energy absorption associated with high wavenumbers damping, in this case.

We interpret this regime as generic, associated with wave spectrum discreteness due to periodicity of boundary conditions. The characteristic feature of this regime is formation of ring structures around $\vec{k} = 0$ (see Fig. 2).

Frozen Turbulence

The numerical experiments described above have demonstrated that the theory of weak turbulence is correct in two-dimensional case, as well as existence of the Kolmogorov spectrum. This result is confirmed by data of laboratory experiments carried out in Department of Physics, UCLA [2].

Still, there are several questions to be answered:

1. Difference of the experimental value of Kolmogorov constant from the theoretical one.
2. Existence of fluxless or "frozen" turbulence regimes at very low levels of short-wave forcing.
3. "Wedding cake" shape of the "frozen" turbulence spectrum (Fig.2) which gives oscillating one-dimensional spectrum after angle-averaging.

4. Maps of quasi-resonances

To understand the answers to above questions we propose the following algorithm of studying of quasi-resonances.

For three vectors

$$\vec{k_1} = (k_{1x}, k_{1y})$$
$$\vec{k_2} = (k_{2x}, k_{2y})$$
$$\vec{k_3} = (k_{3x}, k_{3y})$$

conservation laws (5)–(7) transform into

$$(k_{1x}^2 + k_{1y}^2)^{\frac{3}{4}} + (k_{2x}^2 + k_{2y}^2)^{\frac{3}{4}} - ((k_{1x} + k_{2x})^2 + (k_{1y} + k_{2y})^2)^{\frac{3}{4}} = \Delta_{k_1 k_2}$$

We are building the "map" function $M_\epsilon(k_x, k_y)$ such that

$$M_\epsilon(\vec{k}) = \begin{cases} 1 & \text{if } \Delta_{k_1 k_2} \leq \epsilon \\ 0 & \text{if } \Delta_{k_1 k_2} > \epsilon \end{cases}$$

Every map $M_\epsilon(k_x, k_y)$ corresponds to the chosen "level" of the turbulence ϵ. The algorithm of building of the map tests if every allowed triplet $\vec{k_1}, \vec{k_2}, \vec{k_3}$ has a discrepancy $\Delta_{k_1 k_2}$ less then ϵ. If the answer is "yes" all points $\vec{k_1}, \vec{k_2}, \vec{k_3}$ are assigned the value of 1, and 0 otherwise.

It is important to note that, generally speaking, a particular map is also a function of the cutoff wavenumber in Fourier space k_{cut} which is characteristic value of starting of significant high-wavenumber damping. The more is k_{cut}, the more active resonances exist on the map. This is clear from the following consideration. Suppose that absolute value of $\vec{k_1}$ is much smaller than $\vec{k_2}$, $\vec{k_3}$. It is clear that the bigger k_{cut} is, the more possibilities exist to satisfy the condition $\Delta < \epsilon$ for any given ϵ.

Fig. 3a, 3b, 3c show the maps of quasiresonances for $\epsilon = 0.0001$, $\epsilon = 0.01$, $\epsilon = 1.0$. White areas correspond to "allowed" Fourier modes while blacks to "prohibited" ones. As one could expect, the richness of resonances grows significantly with ϵ. The picture of resonances on Fig.3a is very poor - direct analysis shows that there are no two different triplets $\vec{k_1}, \vec{k_2}, \vec{k_3}$ coupling with each other. This case corresponds to the case of pure "frozen" turbulence, because there is no mechanism of energy transfer from one triplet to another, i.e. from low to high wavenumbers.

The picture of resonances on Fig.3b is significantly denser. It was found that there are coupling triplets of wavevectors in this case able to transfer the energy from low to high wavenumbers, still not many. It is interesting that averaging the map over the angle in Fourier space we get oscillatory one-dimensional "wave spectrum" due to the presence of spectral holes on the corresponding two-dimensional map. It is tempting, but hard to compare these oscillations with low-wavenumber oscillations in laboratory data [2], where some of them are obviously produced by effects of parametric forcing. Still, the tendency of formation of oscillatory spectrum was quite obvious in numerical experiments (see Fig.2) and represents an interesting subject of investigation in laboratory experiments.

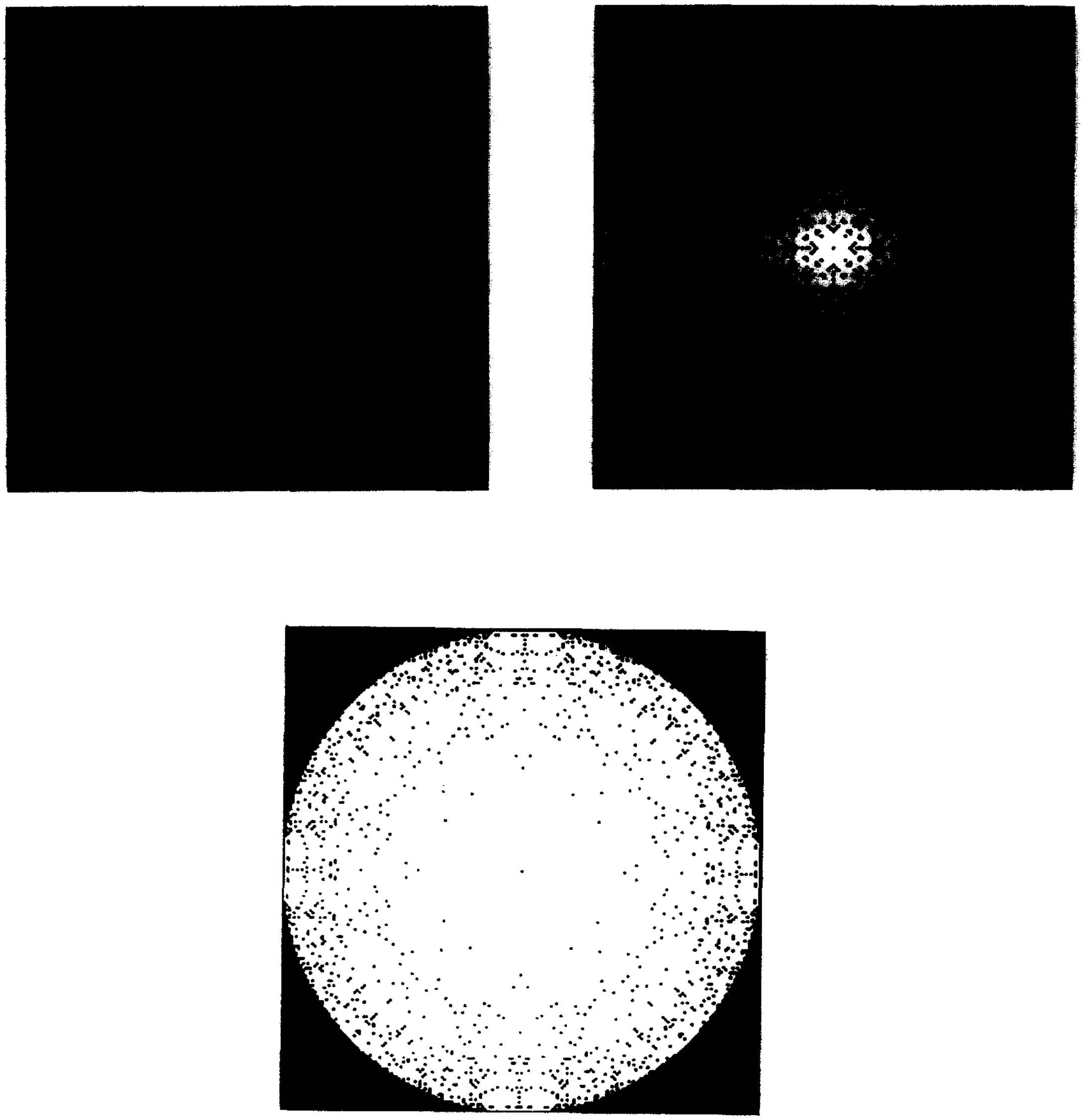

FIGURE 3. Map of quasi-resonances $M_\epsilon(k_x, k_y)$ for (a) $\epsilon = 0.0001$ (b) $\epsilon = 0.01$, (c) $\epsilon = 1.0$. The point $k_x = 0$, $k_y = 0$ is located at the center of the picture. White areas correspond to 1 ("allowed" modes), black area to 0 ("prohibited" modes).

The map of resonances on Fig.3c presents the case of well-developped coupling of resonant triplets. The result of the averaging of the map over the angle does not contain any oscillations. One can expect that the effects of "frozen" turbulence should be minimal in this case being compared to the cases Fig.3a,b which creates better conditions for realization of Kolmogorov regime of turbulence.

5. Conclusion

We performed numerical simulation of weakly turbulent capillary waves which clearly shows existence of Kolmogorov law of the stationary spectrum with the index corresponding to found theoretically from kinetic equation for waves. The value of Kolmogorov constant measured from numerical experiments is happened, however, to be different from the theoretical value.

This difference is caused by the fact that beside Kolmogorov regime there is another, "frozen" regime of turbulence without energy flux from low to high wavenumbers. This mechanism is especially robust at very low levels of turbulence and is observed through anglular symmetric "wedding cake" shaped spectra of turbulence.

The mechanism of "frozen" turbulence can be understood through the analysis of solutions of kinematic three-wave quasi-conservation laws. This analisys was performed numerically by building the maps of quasi-resonanses which show that for small levels of excited waves Fourier space is splitted to the regions of "allowed" and "prohibited" modes. As a result, for very small levels of excitation, there are no coupling triplets of the wavevectors responsible for energy transfer from low to high wavenumbers. The number of "allowed" Fourier modes grows significantly with the increase of the level of excitation or inertial range in Fourier space. As a result, one can expect Kolmogorov regime of turbulence at relatively high levels of excited waves and big enough inertial range in Fourier space. Weak turbulence in bounded systems is therefore, as a rule, the mixture of "frozen" and Kolmogorov turbulence.

The smallness of the experimental value of the Kolmogorov constant suggests that energy flux realized in the numerical experiment is significantly bigger than weak-turbulent flux which could correspond to this constant. Excessive energy flux is, to our mind, caused by narrowness of the intertial range in Fourier space, which makes possible direct (non-cascade) absorbtion of the energy produced in the short wavenumbers region of the pumping.

It is the challenge to build a simplified dynamical numerical model of purely "frozen" turbulence. It can be based on a numerical algorithm which consists of the solution of dynamical equations coupled with dynamically changing map of allowed modes in time and Fourier space. It is also tempting to observe such "frozen" turbulence in laboratory experiments on excitation of capillary waves in containers which could be observed via detection of ring structures in two-dimensional spectra of surface elevations.

Acknowledgments

This work was supported by the Office of Naval Research of the USA (grant ONR N000 14-98-1-0070) and partially by Russian Basic Research Foundation (grant 94-01-00898). We are using this opportunity to acknowledge these foundations.

Thanks are due to Computational Mechanics Publications for their kind permission to reproduce some parts of this paper which first appeared in [4].

References

[1] Zakharov, V.E., Falkovich G., Lvov V.S, Kolmogorov Spectra of Turbulence I, Springer-Verlag, Berlin, 1992

[2] W.Wright, Raffi Budakian, S.Putterman, Phys.Rew.Lett, V.76, N 24 (1996)

[3] Kartashova E., Wave Resonances in Systems with Discrete Spectra, in: V.E.Zakharov (Ed.), Nonlinear Waves and Weak Turbulence, Advances in the Mathematical Sciences (formerly Advances in Soviet Mathematics), American Mathematical Society, Providence, Rhode Island, 1998, pp.95-129

[4] Pushkarev A.N., Zakharov V.E., Turbulence of Capillary waves - theory and numerical simulation. Nonlinear Ocean Waves, Chapter 4, in: W.Perrie (Ed.), Advances in Fluid Mechanics, Vol. 17, Computational Mechanics Publications, Southampton, Boston, 1998, pp.111-131

[5] Zakharov V.E., Filonenko N.N., The energy spectrum for stochastic oscillation of a fluid surface, Doklady Akademii Nauk SSSR, 170(6), (1996) pp.1292-1295, (in Russian)

[6] Zakharov V.E., Filonenko N.N., Weak turbulence of capillary waves, Journal of Applied Mechanics and Technical Physics, 4, (1967), pp.506-515